Water storage and economic resilience in a managed hydrologic system

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Abstract

Resilience and stability are often identified as valuable characteristics of economic, social, and natural systems. Various factors contribute to system resilience and stability, including system complexity, complementary and substitutability, resource stocks and storage. This paper examines the relationship between storage capacity and system stability in a managed hydrologic system in which an manager maximizes the value of stochastic inflows using available storage capacity to affect outflows over time. We first develop theoretical foundations for optimal storage utilization subject to stochastic inflows, limited storage capacity, and an expected utility maximization objective. We then adopt two metrics relating to system stability that are calculable based on estimation results from a Vector Auto-Regression (VAR) framework. In this context, *Reactance* describes the scale of initial reaction of optimized outflows to an inflow shock (e.g. a drought or heavy precipitation), and *resilience* describes the rate of return to steady-state outflows following an inflow shock. Based on this framework we simulate how water storage capacity constraints and preferences over intertemporal water allocation affect system stability as measured by outflow variance, reactance, and we estimate the economics value of these metrics in the context of the hydroeconomic system.

Keywords: Stability, reactance, resilience, stochastic inflows, outflows, storage capacity, change in storage, managed hydrologic system, eigenvalue, steady-state equilibrium

1 Introduction

Resilience and stability are often identified as important and valuable characteristics of economic, social, and natural systems. In agricultural systems for example, yield stability is a desirable system property (Sileshi et al., 2012; Abbo et al., 2010), since large variations in yield can have impactful income and food security consequences. Beddington et al. (1976) and Berardi et al. (2011) identify resilience and stability as critical properties of general dynamic systems, and are quantifiable measures of a system's ability to respond to stochastic variation and persist in stochastic dynamic environments.

Natural hydrologic systems such as watersheds are complex and often characterized by large stochastic variability in inflows through precipitation, temperature, and percolation. Artificial water storage and control rules are often implemented to manage temporal variability in water supply to smooth outflows. For instance, reservoirs in a river can be used to store water between the spring freshet and the irrigation season, or to help contain floods and save water for drought. In this way, storage serves two related roles: It can be used to transfer resources from one time period to others in response to intertemporal preferences, and to buffer against extreme occurrences such as floods and droughts. In other words, water storage can be used to increase stability in stochastic dynamic systems.

Relative storage capacity constraints limit how effective water storage can be used to manage inflow fluctuations. For example, Graf (1999) studied impacts of water storage on river discharge in the continental U.S. states, and finds that in the Great plains, Rocky Mountains and the arid Southwest where large reservoirs hold up to 3.8 times the mean annual runoff, impacts on river discharge are significantly greater than other regions (Northeast and Northwest regions), where storage is only up to 25% of the mean annual runoff. The ratio of reservoir capacity to mean annual flow is termed the Impounded Runoff Ratio (IBI; Batalla et al., 2004), and has been shown to be major determinant of the effectiveness of dams and reservoirs in regulating downstream flow (Kondolf and Batalla, 2005). Reservoirs with low IBI (<0.5) are limited in their ability to regulate flow, and in turn managing system reactance and resilience. Importantly, the size of the impoundment, expressed hydraulically or otherwise, only captures the potential to manage flow; empirically, it is observed that flow management effectiveness is just as dependent on the operational decision making or spill control rules implemented with each reservoir (Katz & Luff, 2020). Thus, forecasting the values added by deploying large storage is difficult and may provide limited marginal value for managing system stability.

A system's stability can be decomposed into at least two components: *reactance* and *resilience*. Reactance reflects a system's tendency to deviate from equilibrium or steady-state in immediate response to an exogenous disturbance (Ives et al., 2003; Neubert and Caswell, 1997), and is analytically homologous to damping in a vibrating mechanical system. Resilience, or *reciprocal return time*, represents the speed with which a system returns to its original, pre-disturbance position or state (Holling, 1973; Webster et al., 1975; Pimm, 1979; Neubert and Caswell, 1997). In a water system in which a manager uses storage to manage outflows over time in response to stochastic inflows, reactance would reflect the size of initial deviations in outflows (as a measure of system performance) from their equilibrium immediately following an inflow shock. Resilience describes the rate at which outflows return to steady state after a one-time inflow shock.

The objective of this paper is to investigate how active water storage management is translated into system stability as measured by reactance, resilience, and related stability measures of economic interest. We construct a theoretical model of a simple managed water system with stochastic inflows and limited storage capacity. We link this theoretical model to a vector auto-regression (VAR) model from which reactance and resilience metrics can be defined and extracted. We then develop a simulation model to examine the effect of storage capacity on system reactance, resilience and related stability measures, and how the economic value of the managed resource relates to these measures.

We begin with a water balance or "budget" equation describing the inter-period flow of water in and out of a system given stochastic inflows and storage capacity to manage outflows (Feiring et al., 1998). A storage manager maximizes the present expected discounted utility of outflows for all periods into the future. We show that both the optimal storage policy rule and its induced optimal outflow path depend on the distribution of inflows, storage capacity, and time and outflow preferences for managing storage.

The joint evolution of the optimal storage and outflows can be characterized by a structural vector auto-regression (SVAR) model, the stability characteristics of which are embodied in the Jacobian Matrix of a VAR, which relates current states of the system with past states. We adopt stability measures developed in the community ecology literature (Ives, 1995; Neubert et al., 2009), and apply them in the hydrology context. Specifically therefore, *Reactance* describes the scale of initial response of optimized outflows to an inflow shock such as a drought or heavy precipitation, and *resilience* describes the rate of the system's return to steady-state outflows following an inflow shock. Together, these measures describe what can be conceived of a system-level impulse response function like those commonly constructed in economic applications of VARs.

We illustrate the implications of our model with a set of simulations. Given stochastic water inflows and managed storage, we show that a water system's reactance (short-term response) is affected by storage capacity relative to the distribution of inflows. In comparison, resilience is affected by both storage capacity and the discount factor with which storage is managed. Greater system stability is observed when the storage capacity is appropriately large and managed with a low discount factor. Second, we verify that lower variability in outflows provide greater utility (or satisfaction) from outflow use, assuming outflow utility is of a quadratic form. In addition, we show the value of storage for providing smooth outflows is subject to the law if diminishing marginal returns. Finally, we identify that nearly 95% of the value of water storage for achieving stability in outflows can be attributed to reduction in reactance.

Our findings contribute to the literature on water resource management and system stability of both the economics and ecology literature. Previous economics studies on the impacts of hydrologic-fluctuations and extremes on economic growth are numerous (Barrios et al., 2010; Sadoff et al., 2015; Borgomeo et al., 2018; Brown and Lall, 2006). These studies find that rainfall variability significantly impacts GDP on local, regional and global scales. Perhaps the closest study to ours is that by (Brown and Lall, 2006), who study the role of water storage in relationship between rainfall variability and GDP for 163 countries. They propose a water resources index that specifies storage infrastructure requirements needed to reduce rainfall variability impacts on water availability for economic activities in different countries. Like Brown and Lall (2006), we illustrate the role of storage capacity for dampening horologic variability, but we dissect system response into two important component parts, and examine the economic value of water storage through these component parts of system stability in a minimally complex, abstract context.

Applying the concepts of stability, resilience and reactance from the fields of ecology and water resource engineering to study a water system in an economics framework is one of our most significant contributions. We show that the idea and measurement of resilience in the ecology literature is closely related to impulse response functions in economics. The growing coherence between non-economics approaches and traditional economics ideas for tackling water resource issues as noted by Jain and Singh (2003) and Heinz et al. (2007), makes it useful to synthesize ideas from these fields to provide richer insights for designing strategies for managing dynamic natural resource systems.

Finally, our study highlights how resilience concepts, measurement, and interpretation may differ between modeled ecological systems and economic systems that explicitly incorporate an optimization process. First, In most modeled ecological systems, a rapid rate of return to a steady-state is taken to reflect a resilient system, which is in turn generally (perhaps implicitly) presumed to be a positive attribute. In contrast, optimization under standard assumptions of expected utility maximization given diminishing marginal utility implies an outflow smoothing process that induces a system-level slower return to steady state when storage is available. Resilience can be nurtured through policies and measures that allow a system to better withstand external disturbances (Briguglio et al., 2009), but interpretation of the resilience metric in particular may be qualitatively inverse in managed systems relative to systems more similar to those typically encountered in ecology. Second, in non-economic systems, shocks themselves are often implicitly prejudged as bad. However, economic agents may in fact benefit from system shocks, even if they are risk averse. A water system in which the marginal utility of water is high for average inflows can benefit from unusually high inflows, especially if there is sufficient storage to capture runoff for use in a later periods when it is more valuable. In such situations where a system may benefit from a shock, quick recovery as emphasized in the study of non-economic systems is not necessarily always the most beneficial outcome. Recognizing these fundamental differences can be important for understanding resilience in both managed economic systems and unmanaged systems such as those modelled in the ecological literature.

The remainder of this paper is organized as follows. In section 2 we introduce a water “budget" equation describing the flow of water in and out of a system, and develop a dynamic stochastic programming model for optimal water use by managing storage to control outflows for consumption over time in section 3. We then construct a structural Vector Auto-regression (SVAR) model using endogenous functions of the optimal water storage and outflows policy rules in Section 4, and proceed to derive measures of resilience and reactance from the (SVAR) model. In section 5, we provide, implement, and discuss results from a numerical example of our model, and conclude in section 6.

2 The water budget

Consider the following features of a simple water system defined by inflows, storage in a single reservoir, and outflows all conceptualized on a seasonal annual timeline. Storage is held in a reservoir for the option to use it next year. Rain falls in the spring, water is used for irrigation in the summer, and unused water from inflows and last year's storage is stored over the fall and winter This flow pattern is depicted in Figure 1.

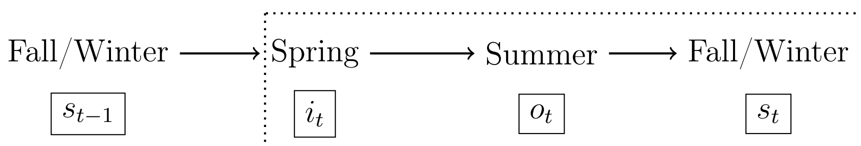

Figure 1: Timing of storage, inflows, and outflows.

Outflows o_t provide value in use, and storage provides value indirectly by allowing available water to be distributed across periods according to decision maker preferences. From an annual perspective, o_t and s_t are jointly determined conditional on predetermined s_{t-1} and i_t, which are both known at outflow decision time. The flow of water in and out of a reservoir system can be described with water balance equation (Feiring et al., 1998)

$$s_{t-1} + i_t = o_t + s_t. \tag{1}$$

The water balance Equation (1) can be rearranged as

$$o_t = i_t - \Delta s_t, \tag{2}$$

where $\Delta s_t = s_t - s_{t-1}$ is the change in the amount of water in storage between periods.[1]

The quantity of water held in storage must be non-negative, and is constrained by a maximum storage capacity $\bar{s}$. Define $\bar{\Delta} s_t = (\bar{s} - s_{t-1})$ as the remaining excess storage available at time t (that is, carry-over storage capacity from the last outflow/storage decision). The storage constraints $s_t \in (0, \bar{s})$ imply the following constraints in Δs_t:

$$- s_{t-1} \leq \Delta s_t \leq \bar{\Delta} s_t, \tag{3}$$

which is to say that storage cannot be reduced by more than the amount available in storage, and cannot be increased by more than the space available given last period's storage level.

Assume that inflows follow an intertemporally independent stationary stochastic distribution $i_t \overset{iid}{\sim} (\mu, \sigma^2)$ where μ and σ^2 are its mean and variance respectively. Inflows can then be characterized as

$$i_t = \mu + v_t, \tag{4}$$

where $v \overset{iid}{\sim} (0, \sigma^2)$ is random variation around the mean. Substituting the right side of Equation 4 into Equation 2 provides

$$o_t = \mu + (v_t - \Delta s_t). \tag{5}$$

Equation 5 shows that outflows depend on mean inflows, inflow deviations from the mean, and the change in storage.

3 Outflow goals and storage requirements

Storage can be used to "move" water release over time by holding inflows to be released from storage for later use, perhaps because the marginal value of water is higher at a later time. It can also be used to reduce the variance in outflows subject to variable inflows and a storage constraint, if this is consistent with management objectives. This paper focuses on the former use of storage.

We develop a general expected utility maximization model and derive optimal policy rules for storage and outflows subject to a storage constraint, and the stationary inflow distribution. We

[1]We omit evaporation and other loss for simplicity.

then develop a structural vector Auto-regression (SVAR) model using the joint evolution of the two policy functions and conduct simulations to show the relationship between storage constraints and outflow outcomes and the measures of system stability (reactance/resilience).

3.1 Outflow utility maximization

Suppose a social planner chooses a state-contingent outflow plan that maximizes the discounted expected value of outflows into the future. The social planner's optimization problem is to choose the optimal outflow sequence $\{o_t^*\}_{t\geq 0}$ that maximizes

$$V_0 = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(o_t)\right] \tag{6}$$

subject to

$$o_t = i_t - \Delta s_t, \quad s_t + \Delta s_t \geq 0, \quad \Delta s_t \leq \bar{\Delta} s_t \text{ and } s_0 \text{ given.}$$

$\mathbb{E}[\cdot]$ is the expectation of future outflows conditional on all information available at time t; $\beta \in (0,1)$ is a discount factor that accounts for the agent's preference for present versus future outflows; u is a one-period utility (pay-off) function from outflows, assumed to be smooth, strictly increasing and strictly concave with $\lim_{o\to 0} u'(o) = \infty$ and $\lim_{o\to\infty} u'(o) = 0$. All other variables are the same as described in previous sections.

In recursive or Bellman form (Bellman, 1957; Ljungqvist and Sargent, 2012), the agent's sequential problem in 6 after substituting out the water budget constraint is

$$V_t(\Delta s_t,\ i_t) = \max_{\{\Delta s_t\}} \left[u(i_t - \Delta s_t) + \beta \mathbb{E}_t[V_{t+1}(\Delta s_{t+1},\ i_{t+1} \mid \Delta s_t,\ i_t)]\right] \tag{7}$$

subject to

$$s_t + \Delta s_t \geq 0, \quad \Delta s_t \leq \bar{\Delta} s_t, \ s_0 \text{ given.}$$

In Lagrangian form, Equation 7 can be written as

$$V_t(\Delta s_{t-1},\ i_t) = \max_{\{\Delta s_t, \lambda_t, \gamma_t\}} \left[u(i_t - \Delta s_t) + \beta \mathbb{E}_t\left[V_{t+1}(\Delta s_t,\ i_{t+1} \mid \Delta s_{t-1},\ i_t)\right] + \lambda_t \bar{C} + \gamma_t \underline{C}\right]. \tag{8}$$

$\bar{C} = [\bar{\Delta}s_t - \Delta s_t]$ is the difference between unused storage capacity and additions to storage. $\underset{\sim}{C} = [s_{t-1} + \Delta s_t]$ is the difference between existing storage and withdrawals from storage. The former represents an upper capacity constraint and must be non-negative because net additions to storage cannot be larger than available capacity. The latter represents a storage non-negativity constraint on storage, and must be non-negative because net withdrawals from storage cannot be larger than the amount available to withdraw. In other words. Physically speaking, you cannot add water to a full storage vessel, and you cannot withdraw more water from storage than there is. Lagrange Multipliers λ_t and γ_t are associated with the upper and lower bounds on storage changes, respectively.

The Lagrangian value function equation 8 provides the maximal value that can be obtained from a given state, defined by the pair $(\Delta s, i)$, after considering all feasible policy actions. The feasible sequence of storage levels and their corresponding induced sequence of outflows that satisfy the value function are optimal. Following Wales and Woodland (1983), the Kuhn-Tucker (KT) necessary and sufficient conditions required to obtain a solution for the problem in 8 are

$$u'(i_t - \Delta s_t) - \beta \mathbb{E}_t u'(i_{t+1} - \Delta s_{t+1}) + \lambda_t - \gamma_t \leqq 0 \leqq \bar{C} \tag{9}$$

$$-\bar{C} \leqq 0 \leqq \lambda_t \tag{10}$$

$$-\underset{\sim}{C} \leqq 0 \leqq \gamma_t. \tag{11}$$

The double inequalities in conditions 9 through 11 imply that the two terms on either side of zero, when multiplied together gives zero. In 9, the first term is an inter-temporal equilibrium condition, obtained by taking the derivative of 8 with respect to Δs_t, and applying an envelope condition (Maliar and Maliar, 2013) to arrive at that expression (see appendix A for full derivation). The optimized Lagrange multiplier λ_t represent the marginal value of additional storage, and optimized γ_t represents the cost of the constraint on storage overdraft.

In any given period t, only one of three possible storage outcomes can occur given optimized outflows: (i) storage between zero and $\bar{s}$, (ii) storage equal to capacity $\bar{s}$, and (iii) zero remaining storage, 2). Below, we derive the optimality conditions that lead to each respective outcome.

Case i: Neither storage constraint is binding. If the optimal storage is between zero and the maximum storage capacity ($0 < s_t^* < \bar{s}$ or $-s_{t-1} < \Delta s_t^* < \bar{\Delta} s_t$), neither storage constraint is binding and $\lambda_t = \gamma_t = 0$. KT conditions (9, 10, 11) that Δs_t^* must satisfy the inter-temporal equilibrium condition

$$u'(i_t - \Delta s_t) = \beta \mathbb{E}_t u'(i_{t+1} - \Delta s_{t+1}). \tag{12}$$

The equilibrium condition (12) is an arbitrage condition that must hold whenever preferred outflows o_t^* are such that the associated change in storage is not constrained by storage capacity or storage availability, and next period storage falls between zero and storage capacity: $s_{t+1}^* = s_t + i_t - o_t^* \in (0, \bar{s})$.

Case ii: Storage capacity $\bar{s}$ is binding. If at time t the manager would prefer and would use more storage capacity if it were available, optimal constrained storage at time t would be at capacity $\bar{s}$, and $\Delta s_t^* = \bar{\Delta} s_t$), with $\lambda_t > 0$ and $\gamma_t = 0$. From the KT conditions (9, 10, 11), the optimal storage $\Delta s_t^* = \bar{\Delta} s_t$, satisfies the following condition

$$u'(i_t - \Delta s_t) = \beta \mathbb{E}_t u'(i_{t+1} - \Delta s_{t+1}) - \lambda_t. \tag{13}$$

The intuition for 13 is that whenever the upper limit constraint on storage binding, some amount water or runoff goes uncaptured, constituting extra spill. Having extra spills (outflows) imply lower marginal utility of outflows (lower by $-\lambda_t$). The value of additional storage capacity λ_t (the cost of the storage constraint) is positive in this case.

Case iii: Non-negativity constraint on storage is binding. When current storage and inflows are low, the manager may want to release all available storage and current inflows to maximize current outflows, and might like to release more if more were available. Under this condition of current scarcity, the non-negativity constraint on storage would be binding: $s_t + \Delta s_t^* = 0$; and and $\lambda_t = 0$ and $\gamma_t > 0$. The associated KT condition is

$$u'(i_t - \Delta s_t) = \beta \mathbb{E}_t u'(i_{t+1} - \Delta s_{t+1}) + \gamma_t. \tag{14}$$

In Equation 14, the marginal utility of present outflows will exceed the marginal utility of future outflows (by γ_t) when the current water endowment of storage plus inflows is low, and the cost of being unable to borrow water from future use is positive.

3.2 Storage and outflow policy rules for quadratic utility of outflows

We now introduce a specific utility function to illustrate the decision space more concretely and to provide the specificity necessary to implement useful simulations. Following the lead of Homayounfar et al. (2015) and Khadem et al. (2020), assume the following quadratic utility function for outflows:

$$u(o_t) = o_t - \frac{1}{2\hat{o}} o_t^2, \quad \hat{o} > 0, o_t \geq 0. \tag{15}$$

This form implies diminishing marginal utility and risk aversion in outflows ($u'(o) = (1 - \frac{1}{\hat{o}} o) > 0, u''(o) = -\frac{1}{\hat{o}} < 0$). Utility is maximized at $o_t = \hat{o}$, meaning that more outflows are preferred up to $\hat{o}$, beyond which additional outflows provides diminishing value. Utility remains positive up to $2\hat{o}$, at which point utility associated with additional outflows is negative. diminishing marginal utility could be interpreted as an overabundance relative to the ideal amount, and the negative utility range would be analogous to destructive or costly flooding that reduces utility to the point that having none would be preferred.

Applying the utility function in 15 to the equilibrium conditions 12, 13, and 14 and letting $\delta = (1 - \beta) \in (0, 1)$ represent one minus the discount factor for brevity provides the optimal policy rules for storage and its induced optimal outflow function, respectively (details in Appendix B):

$$s_t^* = \begin{cases} s_{t-1} + v_t - \delta\left(\hat{o} - \mu\right) & \text{if} \quad 0 \leq s_t^* \leq \bar{s} \\ \bar{s} & \text{if} \quad s_{t-1} + v_t > \bar{s} + \delta\left(\hat{o} - \mu\right) \\ 0 & \text{if} \quad s_{t-1} + v_t < \delta\left(\hat{o} - \mu\right), \end{cases} \tag{16}$$

or equivalently in terms of changes in storage,

$$\Delta s_t^* = \begin{cases} v_t - \delta(\hat{o} - \mu) & \text{if} \quad 0 \leq s_t^* \leq \bar{s} \Leftrightarrow -s_{t-1} \leq \Delta s_t^* \leq \bar{\Delta} s_t \\ \bar{\Delta} s & \text{if} \quad v_t > \bar{s} + \delta\,(\hat{o} - \mu) \\ -s_{t-1} & \text{if} \quad v_t < \delta\,(\hat{o} - \mu)\,, \end{cases} \tag{17}$$

Optimal outflows can be calculated by substituting optimal storage as

$$o_t^* = \mu + (v_t - \Delta s_t^*) \quad \text{or equivalently} \quad o_t^* = \mu + z_t, \tag{18}$$

where $z_t \equiv (v_t - \Delta s_t^*)$ is the stochastic part of outflows after subtracting the chosen storage change.

Consider the policy rules for changing storage (Δs_t^*) in Equation 17, beginning with a special case. Suppose the mean inflow μ by chance exactly equals the outflow satiation point $\hat{o}$ defined by the utility function, and suppose also that for the current period of interest, storage constraints are non-binding such that the first line of Equation 17 applies. In this special case, $\delta(\hat{o} - \mu) = 0$, and therefore $\Delta s_t^* = v_t$. The optimal storage change will exactly equal inflow deviation for that period. Further, if storage constraints happen to never bind, optimal outflows would be constant and equal to μ in each and all periods. This result is not inconsistent with models such as the permanent income hypothesis (Friedman, 1957; Carroll, 1997) and other related models that suggest individual and social preference for consistency in dynamic environments (Grant et al., 2000), and illustrates the a fundamental preference for lower variance in consumption that follows from diminishing marginal returns to consumption.

If in contrast $\hat{o} > \mu$ such that mean inflows are lower than preferred and water is scarce on average, and storage constraints are again nonbinding, Δs_t^* will be lower than v_t (by $\delta(\hat{o} - \mu) > 0$) due to time preferences (represented by $\delta = (1 - \beta)$), and over time there will be a drift in s_t^* toward the lower storage constraint $s_t = 0$. If on the other hand $\hat{o} - \mu < 0$ such that there is more water on average than preferred, and storage constraints are non-binding, then Δs_t^* will be slightly higher than v_t because the agent on average prefers less water and discounts the future. As a result, there will be an upward drift in s_t^* toward $\bar{s}$. Any perturbations of v_t large enough away from the nearest storage constraint will offset this drift temporarily, leading to perturbations away from whichever storage constraint (0 or $\bar{s}$) storage choices tend toward.

When storage constraints are binding such that they limit outflows to be either too low or too high relative to unconstrained arbitrage (lines 2 and 3 of equations Equations 16 or 17), the manager simply either releases all available water for current consumption, or uses all available storage to hold as much back as possible.

4 Empirical measures of system Stability

The joint dynamics of outflows o_t^* and water storage s_t^* based on the policy functions shown in 16 and 18 can be empirically represented as a structural vector auto-regression (SVAR) model. Vector Auto-regression (VAR) is the analogue to Multivariate Auto-regression (MAR) in Biostatistics literature (Hampton et al., 2013). This statistical regression-based representation allows indirect estimation of well-defined measures of system stability in the form of resilience and reactance metrics.

Above, we introduced a specific utility functional form for specificity to support simulations and exposition, and it is now useful to adopt a specific inflow distribution to introduce a statistical regression model to represent a system whose stochasticity comes from inflows. For our purposes it is convenient and reasonable to assume that inflows follow a lognormal distribution (Bowers et al., 2012). To be consistent with the notation used above, let $i_t \sim \text{Lognormal}(\mu, \sigma^2)$ so that $\ln(i_t) \sim \mathcal{N}(\mu, \sigma^2)$, and $v_t \sim \mathcal{N}(0, \sigma^2)$.

The Policy functions in 16 and 18 show that were it not for storage constraints, the relationships between o_t^* and s_t^* and their past values could be represented by a pair of linear (in parameter) regression functions. However, the storage constraints add non-linearity to these relationships. They can nonetheless be approximated to the first-order with the following two variable Structural Vector Autoregression of order 1 (SVAR(1)). Further, water budget Equations 2 and 5 suggests that if inflows are lognormal, variation in outflows will likely be approximately lognormal as well. We therefore apply notation to recognize log-linearity in outflow relationships. The first-order SVAR model that follows from our theoretical model can be written in linear matrix form as

$$\underbrace{\begin{bmatrix} 1 & \alpha_{12} \\ \alpha_{21} & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} s_t^* \\ \ln(o_t^*) \end{bmatrix}}_{\boldsymbol{X}_t} = \underbrace{\begin{bmatrix} \gamma_{10} \\ \gamma_{20} \end{bmatrix}}_{\boldsymbol{\Gamma}_0} + \underbrace{\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}}_{\boldsymbol{\Gamma}_1} \underbrace{\begin{bmatrix} s_{t-1}^* \\ \ln(o_{t-1}^*) \end{bmatrix}}_{\boldsymbol{X}_{t-1}} + \underbrace{\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} v_t \\ z_t \end{bmatrix}}_{\boldsymbol{\varepsilon_t}}, \tag{19}$$

Where $\boldsymbol{X_t}$ is a 2×1 vector of the values of optimal storage and outflows in period time t; $\boldsymbol{X_{t-1}}$ is the observation of $\boldsymbol{X}$ one period back; $\boldsymbol{\Gamma_0}$ is a 2×1 vector of constants; $\boldsymbol{\Gamma_1}$ is a 2×2 time-invariant co-efficient matrix; and $\boldsymbol{\varepsilon_t}$ is a 2×1 vector of structural shocks. The structural matrix $\boldsymbol{A}$ defines the contemporaneous relationship between o_t^* and s_t^*, and $\boldsymbol{B}$ defines the contemporaneous relationship between the structural shocks in $\boldsymbol{\varepsilon_t}$.

To statistically identify the structural components of 19, identification restrictions are typically imposed on the structural matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Hence, for the matrix $\boldsymbol{A}$ we impose the identification restriction that $a_{12} = a_{21} = 0$, meaning shocks to optimal storage will only affect optimal outflows by a lag; and similarly shocks to optimal outflows will only affect optimal storage by a lag. Thus, Shocks do not contemporaneously affect the both optimal storage and outflows.

For matrix $\boldsymbol{B}$, a common identification restriction is to set it equal to the identity matrix (that is, $\boldsymbol{B} = \boldsymbol{I_{2\times 2}}$). We follow this convention and set $\boldsymbol{B}$ equal to the identity matrix so that $\boldsymbol{\varepsilon_t}$ satisfies the following conditions:

1. $E(\boldsymbol{\varepsilon_t}) = \boldsymbol{0}$ - each structural error term has mean zero
2. $E(\boldsymbol{\varepsilon_t \varepsilon_t'}) = \boldsymbol{\Omega}$ - where $\boldsymbol{\Omega}$ is a 2×2 positive semi-definite contemporaneous covariance matrix of the structural shocks (errors)
3. $E(\boldsymbol{\varepsilon_t \varepsilon_{t-k}'}) = \boldsymbol{0}$ for any non-zero k - meaning there is no correlation across time, in particular, no serial correlation in the individual structural error terms.

With above restrictions on $\boldsymbol{A}$ and $\boldsymbol{B}$, system (19) is functionally equivalent to a reduced form VAR(1):

$$\underbrace{\begin{bmatrix} s_t^* \\ \ln(o_t^*) \end{bmatrix}}_{\boldsymbol{X}_t} = \underbrace{\begin{bmatrix} \gamma_{10} \\ \gamma_{20} \end{bmatrix}}_{\boldsymbol{\Gamma}_0} + \underbrace{\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}}_{\boldsymbol{\Gamma}_1} \underbrace{\begin{bmatrix} s_{t-1}^* \\ \ln(o_{t-1}^*) \end{bmatrix}}_{\boldsymbol{X}_{t-1}} + \underbrace{\begin{bmatrix} v_t \\ z_t \end{bmatrix}}_{\boldsymbol{\varepsilon_t}}, \tag{20}$$

or compactly,

$$\boldsymbol{X}_t = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1 \boldsymbol{X}_{t-1} + \boldsymbol{e}_t. \tag{21}$$

The Jacobian matrix $\mathbf{\Gamma}_1$ in system (20) represents the time-invariant relationship between o_t^* and s_t^*, and various articles in the ecology literature derive system stability information from it (Beddington et al., 1976; Ives, 1995; Neubert and Caswell, 1997; Ives et al., 2003; Neubert et al., 2009). Specifically, these studies define system *Resilience* as the largest value of the real eigenvalue $\boldsymbol{\lambda}$ of $\mathbf{\Gamma}_1$ denoted by

$$\rho_1 \equiv \max(\mathrm{Re}(\boldsymbol{\lambda}(\mathbf{\Gamma}_1))). \tag{22}$$

provides information about the asymptotic response of a system to external shocks. Neubert and Caswell (1997) show that the magnitude of the resilience metric ρ_1 reflects how quickly a system recovers following an exogenous shock; quick return (high system resilience) is associated with low ρ_1. The characteristic return time to equilibrium following a shock defined in terms of ρ_1 is (Beddington et al., 1976):

$$t_r = \frac{1}{1 - |\rho_1|}. \tag{23}$$

In 23, the shortest return time is 1, which occurs when $\rho_1 = 0$, meaning a shock is fully absorbed by the system in one period. On the other hand, the longest return time is infinite, which occurs when ρ_1 is large ($\rho_1 = 1$), which corresponds to a random walk. The system is considered unstable when $\rho_1 \geq 1$. A close relationship exist between the above ecological definition of ρ_1 and impulse response functions in economics literature. See the connection between the two in Appendix A.4.

A measure of reactance can be calculated as the largest eigenvalue of the Hermitian (or symmetric) matrix $\boldsymbol{H} = (\mathbf{\Gamma}_1 + \mathbf{\Gamma}_1^{\boldsymbol{T}})/2$, where $\mathbf{\Gamma}_1^{\boldsymbol{T}}$ is the transpose of matrix $\mathbf{\Gamma}_1$ (Neubert and Caswell, 1997). We denote largest eigenvalue–$\boldsymbol{\lambda}^{\boldsymbol{H}}$ of matrix $\boldsymbol{H}$ as

$$\rho_1^H \equiv \max\left(\boldsymbol{\lambda}^{\boldsymbol{H}}(\boldsymbol{H})\right). \tag{24}$$

A system is considered reactive when ρ_1^H is positive, but non-reactive when ρ_1^H is negative (Neubert and Caswell, 1997; Tang and Allesina, 2014). According to Ives (1995), "a highly reactive system tends to move farther away from a stable equilibrium immediately after a perturbation, even though the system will eventually return to the equilibrium point," suggesting that a system with a smaller (less positive) reactivity measure is less immediately responsive to shocks, even if it is, stable.

Together, resilience and reactance metrics ρ_1 and ρ_1^H provide stability information for the system defined in Equation 20. The values ρ_1 and ρ_1^H do not necessarily need to have the same sign. If for example $\rho_1 < 0$ and $\rho_1^H > 0$, the system would be stable but reactive, so that even small shocks will initially be magnified before eventually dying out (Neubert and Caswell, 1997).

5 A Numerical Example

In this section we provide a numerical illustration of our model for two purposes: to illustrate the relationships between water storage capacity and outflow variance, resilience, and reactance; and to estimate the economic value of water storage and changes in reactance, resilience, and outflow variance.

5.1 Implementation

We generate a single time series of 10000 independent and identically distributed inflows (i) from a standard log-normal distribution with mean $\mu_{\ln(i)} = 0$, variance $\sigma^2_{\ln(i)} = 1$.[2] Given the characteristics of the Lognormal distribution, this implies expected value and variance of inflows of $\mu_i \equiv \mathbb{E}[i] = e^{\left(\mu_{\ln(i)} + \frac{1}{2}\sigma^2_{\ln(i)}\right)} = e^{(1/2)} = 1.65$ and $\sigma_i^2 \equiv \text{Var}[i] = 4.67$ respectively (first introduced in Equation 4). The median of this distribution is $m_i = e^{\mu_{ln(i)}} = 1$. The initial storage level s_0 is set to $s_0 = 0$.

We use the quadratic outflow utility specification introduced in Equation 15 that we previously used to illustrate storage rules. The parameter $\hat{o}$ in the utility function is set at $\hat{o} = \frac{1}{2}\mu_i$ or $\hat{o} = \frac{3}{2}\mu_i$ so that on average, inflows are half or one and a half of the ideal amount of $\hat{o}$. Negative utility can occur in any given period if $o_t > 2\hat{o}$ if inflows exceed storage capacity sufficiently. This is more likely when mean inflows μ exceed $\hat{o}$.

The representative agent (water manager, social planner), acts under an expected utility function defined by Equations 6 and 15, with time preference parameter β and risk preference parameter k combinations as follows:

[2]The assumption of independent and identically distributed inflows is not without limitation, since ENSO climate cycles lead to inter-annual precipitation cycle and correlation in inflows, see (Wassmann et al., 2009). Future work could look at the scenario where inflows follow an AR(1) process, for instance.

1. $\beta = 0.95$, $\hat{o} = 1$: Agent is relatively patient and risk averse;

2. $\beta = 0.95$, $\hat{o} = 10$: Agent is relatively patient and risk tolerant;

3. $\beta = 0.5$, $\hat{o} = 1$: Agent is relatively impatient and risk averse;

Recall that utility increases in outflows up to $\hat{o}$, and then decreases. The second-order curvature of this quadratic implies a tight inverse relationship between risk preferences and the most preferred outflow level simply by virtue of the quadratic functional form.

It is important to add that an alternative way to model the discount rather than treating it as exogenous, is to model it as endogenous where β incorporates "visceral" influences in managing storage inter-temporarily. Specifically, visceral influences mean the decision maker's discount factor takes the form $\beta(\boldsymbol{\tau})$ where $\boldsymbol{\tau}$ is a vector of visceral states, example the general nature of water scarcity in an area, the type of water users catered to in an area, the kinds of opportunity costs faced in the period etc. Visceral influences are crucial to intertemporal choice because they can give rise to behaviors that may look extremely impatient or even impulsive, as they increase the attractiveness of certain options (Frederick et al., 2002).

Given that the expected inflow level is fixed for this simulation at $\mu = 1.65$, when $\hat{o} = 1$, so that inflows are on average above preferred outflow levels. In contrast, at $\hat{o} = 10$, average inflows are about 16 percent of preferred outflows, indicating water scarcity on average.

We compute optimal storage (s^*) and outflow (o^*) sequences using Equations 16 and 18 respectively, for twenty different storage capacities $\bar{s} = \mu \times \{x : x \in (1, 2, \ldots, 20)\}$. We present results for the first 200 time periods in the form of time series plots to provide visual illustrations for the distributions of optimal storage s^* and outflows over time o^*.

Using the simulated data on s^* and $ln(o^*)$, we estimate the reduced form (VAR) specification in 20. The eigenvalues representing resilience ρ_1 and reactance ρ_1^H are then obtained from the estimated $\boldsymbol{\Gamma}_1$ matrix for all twenty storage scenarios. Characteristic return times t_r are computed using estimates of ρ_1 in Equation 23.

We then compute the expected present indirect utility of Optimal outflows using the value function:

$$V_0^* = \mathbb{E}\left[\sum_{t=1}^{5000} \beta^t \left(o^* - \frac{1}{2\hat{o}}\, o^{*2}\right)\right], \tag{25}$$

which is obtained by substituting the quadratic utility function in 15 into the objective function in 6. V_0^* represents the total indirect utility (interpretable as economic value) provided by a given sequence of optimal outflow levels.

5.2 Simulation Results

Figure 2 shows simulation results for optimal storage (s^*), storage change (Δs^*), and outflows (o^*) obtained for three different time preference water scarcity combinations ($\beta, \hat{o}$), compared across three storage capacity scenarios with storage capacity of one, five and ten times the expected inflow level μ_i.

The top and middle rows of figure 2 show that regardless of the time and risk preferences by which storage is managed, both optimal storage (s^*) and optimal storage change (Δs^*) are biggest when storage is largest. At the same time, the lower row shows that the outflow variance ($var[o^*]$) decreases towards zero as storage capacity is increased from one to ten times the expected inflow level. The optimality conditions shown in Equations 12, 13 and 14 imply that smooth outflows are preferred, and more storage capacity, $\bar{s}$, allows inflow variation to be more effectively dampened by changing storage.

The last column of figure 2 shows that the impatient agents facing water scarcity ($\beta = 0.5, \hat{o} = 1$) hold the most water in storage on average ($mean[s^*] = 12.66$). In this simulation the agent also faces an overabundance of water ($\hat{o} = 1 < \mu_i = 1.65$), which manifests in holding water back in storage to reduce flows in the current period at the risk of exacerbating an upper storage constraint next period. In contrast, a more patient agent with the same level of risk aversion ($\beta = 0.95, \hat{o} = 1$) will tend to accept more of the overabundant outflows in order to save storage space ($mean[s^*] = 7.84$) to mitigate the risk of future flooding contributed by storage capacity constraints. This patience provides a lower variance in optimal outflows ($var[o^*] = 1.76$) than the impatient agent ($var[o^*] = 3.03$). The patient, risk tolerant, but water-scarce agent holds less in storage on average ($mean[s^*] = 3.31$), but also maintains the lowest outflow variance ($var[o^*] = 1.18$).

Figure 3 shows relationships between standardized storage capacity ($\bar{\bar{s}} \equiv \frac{(\bar{s}-\mu)}{\sigma^2} = \frac{(\bar{s}-1.65)}{4.67}$), and optimal outflow variance, outflow reactance (ρ_1^H), resilience (ρ_1), and return time to steady state (t_r). We use standardized storage levels $\bar{\bar{s}}$ for graphical illustration because it normalizes storage

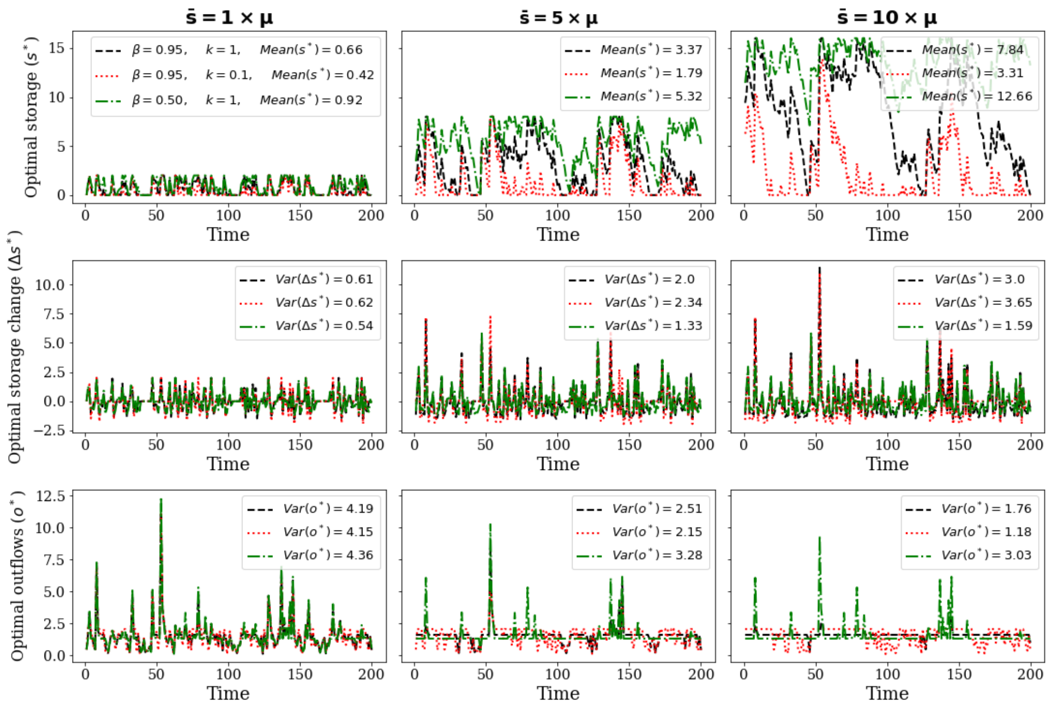

Figure 2: Optimal storage, storage change, and outflow distributions for three storage capacity scenarios. Each distinctly colored dash line depicts a distinct time — risk preference combination (β, k) with which storage is managed.

relative to the inflow distribution for more consistent comparisons. In the top panels of figure 3, smaller values of both optimal outflow variance and reactance (ρ_1) are associated with larger levels of standardized storage capacity ($\bar{\bar{s}}$).

As shown in figure 3 (top left), the fastest decline in optimal outflow variance as standardized storage capacity increases is observed where storage is managed by a patient, risk-tolerant agent ($\beta = 0.95, k = 0.1$) in the scenario where water is scarce ($\hat{o} == 10 > \mu = 1.65$). A patient, risk-tolerant agent tends to smooth outflows over time most effectively such that outflow variance is lowest over all standardized storage capacities. This coincides with the highest reactance as well (upper right), reflecting a willingness and ability to respond to large inflow shocks with correspondingly large changes in storage. in our scenario, the impatient agent ($\beta = 0.5$) is the least effective at reducing outflow variance for a given standardized storage capacity. This is reflected in lower system reactance.

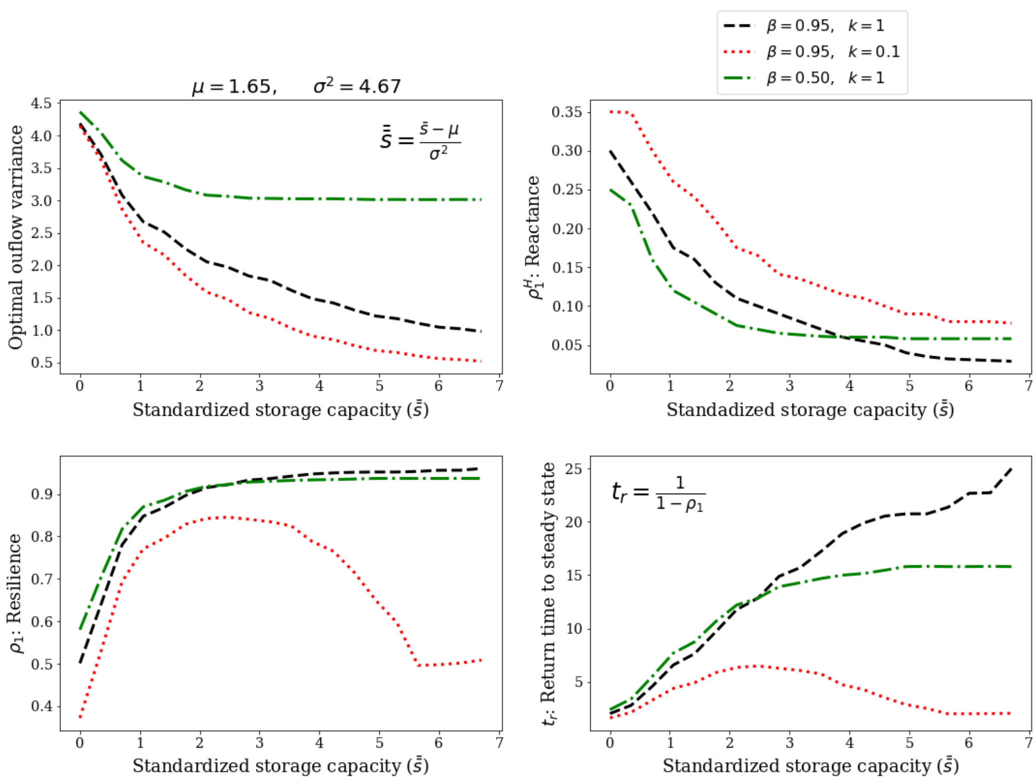

Figure 3: Relationships between standardized storage ($\bar{\bar{s}}$) and optimal outflow variance, reactance (r_1^H), resilience (r_1), and return time to steady state (t_r).

The bottom row of Figure 3 show relationships between standardized storage capacity ($\bar{\bar{s}}$), and resilience (ρ_1) and return time (t_r). The patient, risk-tolerant, water-short agent ($\beta = 0.95, k = 0.1$) creates an inverted "U-shaped" resilience function that then flattens as standardized storage increases, and a short return time to steady-state. Increasing from low standardized storage capacity increases the agent's ability to spread inflow shocks across more periods to minimize outflow shocks in any single period, but only up to a point. Initially, this generates stronger inter-temporal auto-correlation in optimal outflows, and consequently, larger resilience values (ρ_1), and longer return time to steady state. Past the peak of the inverted "U" however, additional increases in storage capacity allow inflow shocks to be dampened quickly (immediately in the case of very large storage capacity), leading to lower inter-temporal auto-correlation in optimal outflows and low measured system resilience (ρ_1) values, and faster return times return time to steady state (t_r). The flat part of the curve suggests that as storage capacity increases beyond a certain point, more storage becomes inconsequential with respect to impacts on resilience and return time to steady state.

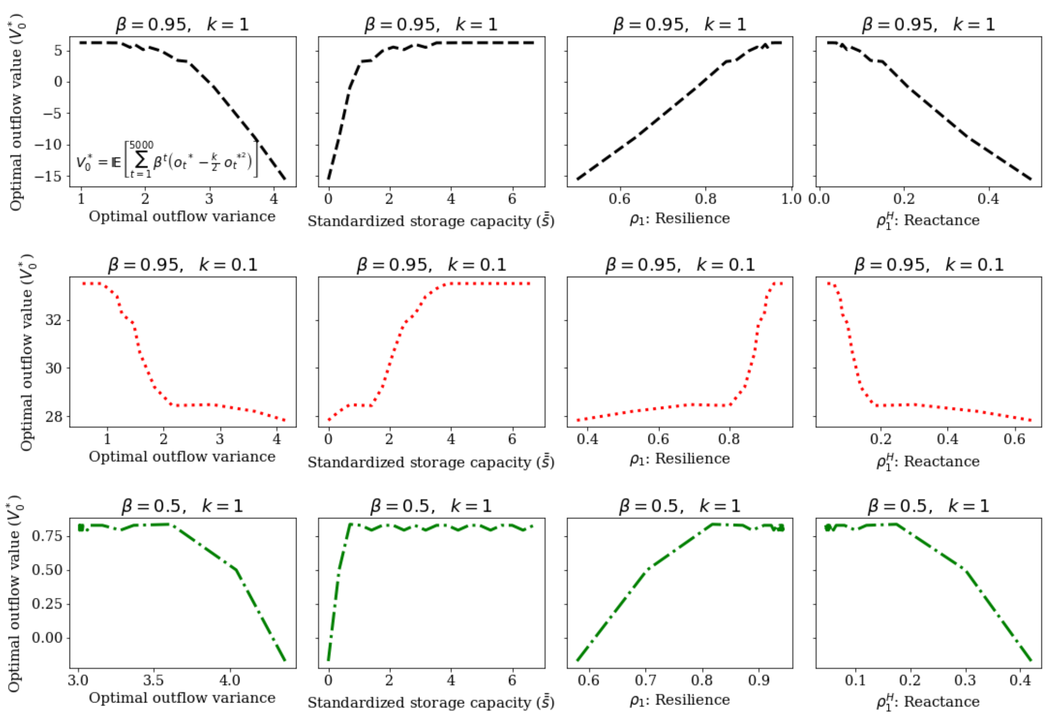

Figure 4: Relationships between optimal outflow value (V_0^*) and outflow variance, standardized storage capacity ($\bar{\bar{s}}$), resilience (ρ_1), and reactance ρ_1^H, compared across three time and risk preference parameterizations.

In the the other two scenarios (black and green curves in the bottom panels of figure 3), both the patient, risk-averse ($\beta = 0.95, k = 1$) agent and the impatient, risk-averse ($\beta = 0.5, k = 1$) agent manage storage in an environment where there is an overabundance of water. An increase in storage capacity only provides a means to accumulate more water and will thus be less flexible in optimally changing storage from period-to period. This rigidity in changing storage as indicated by lower reactance propagates stronger auto-correlation in outflows, which correspond to larger resilience values, and longer return time to steady state. Here, the effect of increases in storage capacity increase resilience, and return time quickly reaches near maximum. Return times continue to increase for the patient agent, but top out at about 15 periods for the impatient one.

Figure 4 illustrates relationships between optimal outflow value (V_0^*) (measured by the present expected indirect utility of optimal outflows) and optimal outflow variance, standardized storage capacity ($\bar{\bar{s}}$), and resilience (ρ_1) and reactance (ρ_1^H), compared across the three time utility parameterizations.

The top row of Figure 4 corresponds to a patient, risk-averse agent. For this agent, optimal outflow variance (leftmost column) is inversely related to optimal outflow value. Expected net present value declines relatively slowly as optimal outflow variance increase. The second column shows that the economic value of storage initially increases rapidly with increases in standardized storage capacity $\bar{\bar{s}}$ and then flattens out as standardized storage capacity continuous to increase for the patient, risk-averse agent, illustrating diminishing returns consistent with Langbein (1959). A positive relationship is observed between resilience (ρ_1) and economic value in the third column, top row graph. In contrast, a negative relationship is observed between reactance (ρ_1^H, rightmost column) and economic value. Larger resilience values in this context are consistent with the preference to spread inflow shocks across more periods, and minimize impacts on a single period where storage capacity may be limited. The relationships between both system resilience and reactance on the one hand and expected net present value on the other are relatively linear over the observed range of resilience for the patient risk-averse agent ($\beta = 0.95, k = 1$).

In the other two cases (patient, risk tolerant & impatient), there are ranges of the system behavior metrics where value is insensitive to changes. The middle row of Figure 4 correspond to a patient, risk-tolerant agent ($\beta = 0.95, k = 0.1$). The left- and right-most columns show steep inverse relationships between optimal outflow value and optimal outflow variance, and reactance at low values, but relative insensitivity of value to changes in system behavior over the rest of their ranges. In contrast, there are positive, though non-linear relationships between both $\bar{\bar{s}}$ and ρ_1, and outflow value. For the water-scarce, patient risk-tolerant agent however (middle row), there are broad regions of value insensitivity for all of the system behavior metrics and narrow threshold range that exhibits rapid change. We interpret this threshold to be associated with the point at which more storage provides little value as a buffer against very low inflows and the zero-storage constraint, and is instead used solely for spreading water at the margin across periods as intertemporal arbitrage. Regardless, this case is similar to the others in that the economic value of storage exhibits diminishing marginal returns after some point, but that point is later for the patient, risk tolerant agent relative to the others.

The bottom row graphs in figure 4 correspond to an impatient, risk tolerant agent ($\beta = 0.5, k = 0.1$) in a water-abundant environment. They exhibit similar relationships as the top row graphs, but with more second-order curvature. Of particular interest is that the expected net present value of outflows increases rapidly with standardized storage capacity until it tops out, at which point additional storage does not provide additional value.

5.3 Discussion

Demand for water is often smoother over time than water availability and supply because inter- and intra-seasonal precipitation variation tends to be larger than demand variation in many parts of the globe (Nestmann and Stelzer, 2007). Too much water in one period followed by too little water in the next is economically unfavorable and costly (Brown and Lall, 2006). Economic activities such as hydropower generation have been noted to be highly responsive to stream flow changes, often caused by fluctuations in river discharge, which depend on precipitation levels (Wilbanks et al., 2008; Karl et al., 2009).[3] Storage is a fundamental system component that acts to delay, buffer, or absorb inflow shocks (Meadows and Wright, 2008). It is valuable for mitigating natural variability and reallocating resources through time (Barnett et al., 2005).

[3] Nash and Gleick (1993) estimated that hydropower generation in the Colorado River Basin changed by about 3 percent for every 1 percent change in stream in flow

Langbein (1959) recognized that the ability to regulate river flow using storage depends on the size of storage relative to the volume of river flow, although size of storage has been defined inconsistently in the literature (Graf, 2005, 2006; Kondolf and Batalla, 2005), and validating empirical analysis is scarce (Katz & Luff, 2020). More recently, Hansen et al. (2011) have shown that water storage is instrumental in providing consistent agricultural production amid weather variations, and increases the chances of successful harvests in periods of precipitation extremes (floods and droughts).

The role of storage for achieving stability in a system is measured by its impacts on lowering variation in outflows, which is dissectable into impacts on statistical measures of variance, reactance and resilience. We find that storage impacts on reactance and resilience depend considerably on time-scales and risk preferences that influence storage management. Preferences determine how storage is optimally varied from period-to-period in a given system, and storage use determines how outflow variation relate to inflow fluctuations. In particular, our simulations show that whereas reactance is bigger where optimal storage variance is greater, optimal outflow variance is smaller. In the same vein, large storage capacity allows storage to be more flexibly manipulated over time to produce greater resilience in the form of quicker recovery in outflows to a steady-state condition. Thus, both the amount of storage and the preferences/decisions governing its use determine how storage capacity affects system stability and its components, reactance and resilience. This result is consistent with the expectations provided by Pahl-Wostl (2007), who forecast that system resilience would be affected not only by the physical and infrastructural characteristics defining the system, but by the governance system acted upon in that system. Ultimately, storage capacity, and the manner in which it managed has implications for both reactance and resilience.

Apart from storage capacity and management, other differences in natural resource endowments and system function may allow some water systems to respond better to unforeseen events than others. For instance, we show in our results that system reactance is considerably lower for systems that have larger water endowments, compared to systems with smaller water endowments. Briguglio et al. (2009) explains that systems that lack the necessary endowments and inherent immunity to external disturbances can be modified by policy-induced actions to boost resilience. Briguglio et al. referred to this form of resilience as *nurtured resilience*, suggesting that resilience and stability can be affected by identifying inherent vulnerabilities in the system and applying infrastructure and policy measures to boost immunity against adversity.

There is value in system stability. Basic economic theory and empirical evidence suggests that people typically (but not always) prefer consistency in consumption. Faced with a series of scholastic inflow endowments, storage provides a means to capture and release inflows in a manner that generates consistency in outflows across periods, increasing expected utility from outflows through time. This relationship between inflows, storage and outflows is structurally analogous to the well-known income saving and spending rule in economics: given expectations about future income (future inflows), people will tend to save and borrow (store or release water) when possible to smooth consumption over time (outflows for use). People tend to save when their income is high and draw down savings when income is low. Like savings, the value of storage therefore depends on how much consumption security it provides.

Although these system-level metrics of behavior have been viewed as different aspects of system stability, they are each distinct, represent different components of response, and may not all increase in value together. For example, in the ecology literature in which this system resilience measure was developed (e.g. Neubert and Caswell, 1997), a slow return time to steady state (large ρ_1) is often interpreted, even if implicitly, as low system resilience. This contrasts to our economically optimized system, in which more storage allows for more flexibility to dampen outflow variation, inducing longer return times and more system resilience to a point. However, systems viewed as stable based on long return times could possibly be slow to demonstrate early warning indicators to impending large-amplitude and unstable state changes (Dakos et al., 2008, 2012; Guttal and Jayaprakash, 2008; Scheffer et al., 2009, 2012). In particular, in systems that are driven by large amplitude external forcing, high resilience may be undesirable. In these cases, slow response times and high resilience would limit system level agility, or the ability to respond or adapt to large amplitude shocks or changes in static forcing (Holling, 1973; Donohue et al., 2016; Goodman, 1975). Consequently, there may not be a monotonic mapping of stability onto desirability or value in every case.

The economic value of reactance and resilience as they relate to a water system's stability are estimable in terms of outflow value. Both storage capacity and preferences surrounding storage management affect the economic values of reactance and resilience. We demonstrate that the economic value associated with reactance is concave where storage management is characterized by patience and risk-aversion, but convex where agents are patient risk-tolerant. Also, the marginal economic value of quick outflow return to steady state (smaller resilience values) is smaller than slow outflow return to steady where outflow smoothing over time is the main goal. As pointed out by (Donohue et al., 2016), this contrast between short term shocks and long term press or changes to equilibrium states may not accurately reflect the salience of the external drivers or shocks to system level behavior. How these trade-offs have been deferentially addressed in the economics, ecological and engineering literatures suggests that this question is ripe for an investment of novel interdisciplinary investigation.

6 Summary and conclusion

This paper studies the relationship between water storage capacity and system stability and resilience in a managed hydrologic system in which an agent maximizes the value of outflows, given stochastic inflows using available storage capacity to affect utilization over time. We develop a theoretical framework that demonstrates how storage capacity is used to manage water system stability as measured by variance in outflows, and two other measures calculable from a Vector Auto-regression (VAR) model; reactance, and resilience. *Reactance* measures the scale of initial reaction of optimized outflows to an inflow shock (e.g. a drought or heavy precipitation), and *resilience* describes the rate of return to steady-state outflows following an inflow shock.

Our simulations illustrated some of fundamental relationships. First, system reactance and resilience are affected by both storage capacity and a decision maker's discount factor and risk preferences, which together affect how storage is used to manage outflows over time. Second, a large storage capacity relative to the stochastic inflow distribution is important for managing storage to achieve low variance in outflows. Third, the economic value of storage is subject to diminishing returns, which is consistent with the findings of (Langbein, 1959). Fourth, low variance in outflows presents higher economic value as measured by the present expected indirect utility. Fifth, at any given storage capacity level, the marginal economic value of reactance and resilience are affected by the risk preferences levels of a decision maker.

The methods used in this paper represent a new economic application of system stability measures developed and interpreted largely in the ecology literature in the context of ecological systems. In these ecological systems, resilience measured as fast return to steady state and reactance as a system's ability to respond to and dampen exogenous shocks to systems are often viewed as positive system characteristics. In a managed system driven by a preference for outflow stability, a capacity for flexible system management often leads to what seems to be conflicting outcomes of weaker system resilience and reactance as measured by these metrics. the takeaway of this comparison is that interpreting system stability with these measures of resilience and reactance must be done carefully, with system fundamentals in mind.

Future extensions of this model could include implementation with real-world data to demonstrate through structural estimation or calibration. With a fully-fledged quantitative model, the framework can be used to answer real-world policy questions.

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Appendices

A.1 The inter-temporal equilibrium condition

Here, we derive the inter-equilibrium condition in 9. The first order condition of 8 with respect Δs_t provides

$$u'(i_t - \Delta s_t) - \beta \mathbb{E} v'(\Delta s_t,\ i_{t+1} \mid \Delta s_{t-1},\ i_t) + \lambda_t = 0 \tag{A.1}$$

To find the term $v'(\Delta s_t \mid \Delta s_{t-1}, i_t)$, we employ the envelope condition technique. This involves differentiating 8 with respect to Δs_t, and iterating the result one period forward i.e.

$$\begin{aligned} v'(\Delta s_{t-1}, i_t) &= u'\left(i_t - ((s_{t+1} - s_{t-1}) + \Delta s_{t-1})\right) \qquad \text{(A.2)} \\ \Rightarrow v'(\Delta s_t, i_{t+1}) &= u'\left(i_{t+1} - ((s_{t+2} - s_t) + \Delta s_t)\right) \\ &= u'(i_{t+1} - \Delta s_{t+1}) \end{aligned}$$

Substitute (A.1) into (14) to arrive at

$$u'(i_t - \Delta s_t) - \beta \mathbb{E} u'(i_{t+1} - \Delta s_{t+1}) + \lambda_t = 0 \tag{A.3}$$

A.2 Optimal storage conditions for quadratic utility

The fundamental equimarginal result, shown in Equation 12 as $u'(i_t - \Delta s_t) = \beta \mathbb{E}_t u'(i_{t+1} - \Delta s_{t+1})$, with the addition of Lagrange Multipliers λ_t or subtraction of γ_t if the outflow/storage decision is constrained by storage capacity $\bar{s}$ or zero, respectively.

For the derivations below, note that marginal utility for quadratic utility defined in Equation 15 is given by $u'(o_t) = (1 - o_t/\hat{o}) = (1 - (i_t - \Delta s_t)/\hat{o})$; $\mathbb{E}(i_t) = \mu, \forall i$ by assumption of the inflow distribution, and $\mathbb{E}(\Delta s_{t+1}) = 0$ as shown in Appendix A.3.

Case i: Neither storage constraint is binding

Substituting the quadratic marginal utility of outflows into 12 provides

$$1-(i_t-\Delta s_t)/\hat{o}=\beta\mathbb{E}_t\left(1-(i_{t+1}-\Delta s_{t+1})/\hat{o}\right). \tag{A.4}$$

substituting $\mathbb{E}_t(\Delta s_{t+1})=0$ and $\mathbb{E}_t(i_{t+1})=\mu$ on the right-hand side of Equation A.4 provides

$$1-(i_t-\Delta s_t)/\hat{o}=\beta(1-\mu/\hat{o}). \tag{A.5}$$

Substituting $\mu+v_t$ in place of i_t and s_t-s_{t-1} in place of Δs_t on the left-hand side of Equation A.5 and solving for s_t provides the optimal choice of storage to hold for period t:

$$s_t^i=s_{t-1}+v_t+(1-\beta)\left(\mu-\hat{o}\right), \tag{A.6}$$

where the i superscript indicates case i, the unconstrained storage choice. Substituting $\delta=(1-\beta)$ and $\hat{o}$ (representing the most favored outflow that which maximizes quadratic utility), provides

$$s_t^i=s_{t-1}+v_t-\delta\left(\hat{o}-\mu\right), \tag{A.7}$$

where s_{t-1} is carryover storage from last period and v_t is this period's deviation from mean inflow as shown for the unconstrained case in Equation 16. Subtracting s_t from both sides provides the unconstrained case shown as s_t^* in Equation 17:

$$\Delta s_t^i=v_t-\delta\left(\hat{o}-\mu\right). \tag{A.8}$$

Since storage constraints are non-binding in this case, the optimality condition in 12 implies that the optimal storage must be an interior solution ($0\leq s_{t+1}^*\leq\bar{s}$).

Case ii: Storage capacity is binding

Optimality condition 13 applies in this case. Substituting the quadratic marginal utility of outflows into Equation 13 provides

$$1 - k(i_t - \Delta s_t) = \beta \mathbb{E}_t \left(1 - (i_{t+1} - \Delta s_{t+1})/\hat{o}\right) - \lambda_t. \tag{A.9}$$

The left-hand-side is the current-period marginal utility of outflows. When storage capacity is binding, outflows are higher than would be chosen if the storage constraint were not binding, so marginal utility is lower than it would be (by the amount $\lambda_t > 0$) due to diminishing returns to outflows.

Applying Substitutions as in Equation A.5 and rearranging, and noting again that the Kuhn-Tucker condition Equation 10 implies $\lambda_t > 0$ when $\bar{s}_t$ is binding, provides

$$\lambda_t = \beta(1 - \mu/\hat{o}) - (1 - (i_t - \Delta s_t)/\hat{o}) > 0. \tag{A.10}$$

Equation A.10 provides the basis for calculating the marginal cost of the storage constraint (equivalently, the marginal value of additional storage capacity) for any period t. Rearranging Equation A.10 and substituting as in Equation A.7, and recognizing that in this case $s_t = \bar{s}$ provides

$$s_t^{ii} = \bar{s} < s_{t-1} + v_t - \delta\left(\hat{o} - \mu\right) = s_t^i, \tag{A.11}$$

where superscript ii on s_t indicates an optimum constrained from above by $\bar{s}$. Equation A.11 implies

$$s_t^{ii} = \bar{s} \quad \text{if} \quad s_{t-1} + v_t > \bar{s} + \delta\left(\hat{o} - \mu\right), \tag{A.12}$$

shown as s_t^* in Equation 16. Alternatively, subtracting s_{t-1} from both sides of Equation A.11 provides

$$\Delta s_t^{ii} = \bar{\Delta} s_t = \bar{s} - s_{t-1} < v_t - \delta\left(\hat{o} - \mu\right), \tag{A.13}$$

and

$$\Delta s_t^{ii} = \bar{\Delta} s_t \quad \text{if} \quad v_t > \bar{\Delta} s_t + \delta\left(\hat{o} - \mu\right), \tag{A.14}$$

shown as Δs_t^* in Equation 17. Intuitively, if the available storage $\bar{\Delta} s_t$ is less than this periods inflow deviation plus a constant reflecting a discounted difference between most preferred outflows and expected inflows, the manager will use all storage, and any additional inflow will spill, providing more than optimal outflows and no storage for future flexibility. Note that if water is scarce on average ($(\hat{o} - \mu) > 0$), the manager will never choose to use all available storage when v_t is negative (less than average inflows).

Case iii: Storage non-negativity constraint is binding

Optimality condition 14 applies in this case. Substituting the quadratic marginal utility of outflows into Equation 14 provides

$$1 - (i_t - \Delta s_t)/\hat{o} = \beta \mathbb{E}_t(1 - (i_{t+1} - \Delta s_{t+1})/\hat{o}) + \gamma_t \tag{A.15}$$

In this case, the constraint is that there is not as much carryover water in storage to satisfy the unconstrained optimality conditions. Outflows are lower than a manager would choose if there were, and the current marginal utility of water is higher than in the unconstrained case, by the amount γ_t. The constraint that no water can be drawn from a facility with zero water in the current period is analogous to a constraint on borrowing water from the future, the past, or another storage facility given current conditions. The Lagrange multiplier γ_t is

$$\gamma_t = (1 - (i_t - \Delta s_t)/\hat{o}) - \beta(1 - \mu/\hat{o}) > 0, \tag{A.16}$$

The (optimized) Lagrange Multiplier γ_t represents the cost of not being able to borrow water from the past, future, or another storage facility (or the value of being able to borrow). In a manner consistent with derivations in case ii,

$$s_t^{iii} = 0 > s_{t-1} + v_t + \delta\left(\hat{o} - \mu\right), \tag{A.17}$$

and

$$s_t^{iii} = 0 \quad \text{if} \quad s_{t-1} + v_t < \delta\left(\hat{o} - \mu\right). \tag{A.18}$$

The change in storage is defined in this case as

$$\Delta s_t^{iii} = 0 - s_{t-1} > v_t + \delta\left(\hat{o} - \mu\right), \tag{A.19}$$

implying

$$\Delta s_t^{iii} = -s_{t-1} \quad \text{if} \quad v_t < \delta\left(\hat{o} - \mu\right), \tag{A.20}$$

shown in Equation 17 as s_t^* for case iii. Storage will more often be driven to zero in a water scarce environment $((\hat{o} - \mu) > 0)$ than in an environment with overabundant water $((\hat{o} - \mu) < 0)$.

A.3 Proof: Infinite sum of storage changes equals 0.

This proof shows that $\mathbb{E}_t[\Delta s_{t+i}] = 0, \quad i = 0, 1, 2, ...$ The sum of outflows to date must equal the sum of inflows to date minus any unreleased storage. So for any time t,

$$\sum_{j=1}^{t} o_j = \sum_{j=1}^{t} i_j - o_t. \tag{C1}$$

Subtracting $\sum_{j=1}^{t-1} o_j$ from $\sum_{j=1}^{t} o_j$ based on Equation C1 provides

$$\begin{aligned} o_t &= i_t - (s_t - s_{t-1}) \\ &= i_t - \Delta s_t. \end{aligned} \tag{C2}$$

If equation C2 holds for any period t it holds for each period $i \leq t$, so

$$\sum_{j=1}^{t} o_j = \sum_{j=1}^{t} i_j - \sum_{j=1}^{t} \Delta s_j \tag{C3}$$

Equating the right hand side of Equations C1 and C3 provides

$$\sum_{j=1}^{t} \Delta s_j = s_t. \tag{C4}$$

Divide through by t to get $\frac{1}{t}\sum_{j=1}^{t} \Delta s_j = \frac{S_t}{t}$. For any finite storage capacity, as t gets large, $\lim_{t\to\infty} \frac{s_t}{t} = 0$, so

$$\lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{t} \Delta s_j = \frac{s_t}{t} = 0 \tag{C5}$$

By the law of large numbers (weak or strong), $\lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{t} \Delta s_j = E[\Delta s_t]$. So $E[\Delta s_t] = 0$

A.4 Impulse Response Function and Largest Eigenvalue of a Matrix Relationship

Consider that the reduced form process in Equation 21 is stationary. We can define the lag operator $\boldsymbol{L}_{2\times 1}$, such that $\boldsymbol{L X_t = X_{t-1}}$ and $\boldsymbol{L}^2\boldsymbol{X_t = X_{t-2}}$, and use this to find the following infinite moving average representation:

$$\boldsymbol{X}_t = \boldsymbol{G}_1\boldsymbol{X}_{t-1} + \boldsymbol{e}_t, \tag{D1}$$

where $\boldsymbol{X_t}$ is assumed to be de-meaned so that the term $\boldsymbol{G}_0$ can be dropped without loss of generality. Applying the lag operator to Equation D1 gives:

$$(\boldsymbol{I_{2\times 1}} - \boldsymbol{G}_1\boldsymbol{L})\boldsymbol{X_t} = \boldsymbol{e}_t \tag{D2}$$

$$\boldsymbol{X_t} = (\boldsymbol{I} - \boldsymbol{G}_1\boldsymbol{L})^{-1}\boldsymbol{e}_t \tag{D3}$$

It can be shown that the infinite sequence $(\boldsymbol{I} - \boldsymbol{G}_1\boldsymbol{L})^{-1} := \boldsymbol{\tau}$ in D3 converges to the largest eigenvalue of $\boldsymbol{\tau}$, which dictates the rate at which impulses in $\boldsymbol{X}_t$ due to $\boldsymbol{e}_t$ dissipate (the bigger the eigenvalue value, the slower the dissipation rate and the smaller the eigenvalue value, the quicker the dissipation rate). A proof of this is presented in the following link `https://yutsumura.com/`

sequence-converges-to-the-largest-eigenvalue-of-a-matrix/. From Equation D3 we get:

$$X_t = (\boldsymbol{G}_1\boldsymbol{L})^0\boldsymbol{e}_t + (\boldsymbol{G}_1\boldsymbol{L})^1\boldsymbol{e}_t + (\boldsymbol{G}_1\boldsymbol{L})^2\boldsymbol{e}_t + (\boldsymbol{G}_1\boldsymbol{L})^3\boldsymbol{e}_t + ...$$
$$= \boldsymbol{e}_t + \boldsymbol{G}_1\boldsymbol{e}_{t-1} + {\boldsymbol{G}_1}^2\boldsymbol{e}_{t-2} + {\boldsymbol{G}_1}^3\boldsymbol{e}_{t-3} + ... \tag{D4}$$

The coefficients $\boldsymbol{G_1^j},\ j = 1, 2, 3, ...$ in Equation D4 give the response of variables in $\boldsymbol{X_t}$ to impulses from the structural shock vector $\boldsymbol{e}_{t-j}$. In particular, we can compute the response of $\boldsymbol{X_t}$ to a shock in period t for the periods t, $t+1$, $t+2$ and $t+j$ respectively as follows: $\frac{\partial \boldsymbol{X_t}}{\partial \boldsymbol{e}_t} = \boldsymbol{G_1^0}$, $\frac{\partial \boldsymbol{X_{t+1}}}{\partial \boldsymbol{e}_t} = \boldsymbol{G_1}$, $\frac{\partial \boldsymbol{X_{t+2}}}{\partial \boldsymbol{e}_t} = \boldsymbol{G_1^2}$ and $\frac{\partial \boldsymbol{X_{t+j}}}{\partial \boldsymbol{e}_t} = \boldsymbol{G_1^j}$. The plot of $\frac{\partial \boldsymbol{X_{t+j}}}{\partial \boldsymbol{e}_t} = \boldsymbol{G_1^j}$ for all $j = 0, 1, 2, ...H$ (where H is the time horizon of the plot) gives the *impulse response function (IRF)* of $\boldsymbol{X}$. Notice that both the IRF of $\boldsymbol{X}$ and the measure of resilience (ρ_1) depend directly on the $\boldsymbol{G_1}$ matrix. However, the difference between the two is that, whereas the IRFs provide visual representations of how individual variables in $\boldsymbol{X}$ would respond to a one time shock in $\boldsymbol{e}$, the largest eigenvalue of the system (also known as the principal component of the system) ρ_1, provides information about the overall responsiveness of the system to shocks.